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LETTER TO THE EDITOR

A completeness relation for the q-analogue coherent states by q-integration

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Abstract. q-integration is defined for the q-oscillator realization of quantum groups. This is used to prove a completeness relation for the q-analogue of the usual coherent states. These states are overcomplete.

The concept of integration is very important in mathematical physics and quantum field theory. Jackson [1-3] introduced the idea of q-differentiation and q-integration in 'basic analysis'. Recently, in the q-harmonic oscillator realization of quantum groups [4-17], several authors have extended the definition of q-differentiation [18-20]. In this letter we use the idea of q-integration in the q-oscillator realization of quantum groups to prove a resolution of unity for the q-analogue coherent states.

For a quantum group, such as $SU_q(2)$, we define for q real

$$[n]_q = [n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}. \tag{1}$$

Note that $[n]_q$ is invariant when q is replaced by $1/q$. If we write $q = \exp(s)$ then since $\sinh(n) \equiv \frac{1}{2}[\exp(n) - \exp(-n)]$, we can write $[n]_q = \sinh(sn/2)/\sinh(1/2)$. In the mathematics literature [1-3, 21-26], one finds $[n]_J \equiv (1-q^n)/(1-q)$. So $q^{(1-n)/2}[n]_J = [n]_q$. We define the q-factorial to be $[n]! \equiv [n][n-1][n-2]\ldots[1]$. Then

$$[n]_q! = [n]! = q^{-(n(n-1))/4}[n]_J!. \tag{2}$$

Exton [25, 26] defines a family of q-exponential functions by

$$e_q^\lambda(z) \equiv \sum_{n=0}^{\infty} \frac{z^n q^{\lambda n(n-1)}}{[n]_J!}. \tag{3}$$

When $\lambda = 0$ and $\lambda = \frac{1}{2}$, one gets the two exponential functions defined by Jackson [2]. For $\lambda = \frac{1}{4}$ one gets an exponential function which is invariant to the transformation $q \to 1/q$.

$$e_q(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{[n]!}. \tag{4}$$

Unlike the case of $\lambda = 0$ where the radius of convergence of the series representation is finite, for $\lambda > 0$ this series representation of $e_q(z)$ converges for all finite values of z independent of the value of q. Note that $[1]_q = 1$ for all values of q and for $n > 1$, $[n] \geqslant n$.

So for all $x \geqslant 0$

$$e_q(x) \leqslant \exp(x). \tag{5}$$

For quantum groups, the q-derivative is defined to be [18–20]

$$\frac{\mathrm{d}}{\mathrm{d}_q x} f(x) = \frac{f({}^{-1/2}x) - f(q^{1/2}x)}{q^{-1/2}x - q^{1/2}x}. \tag{6}$$

For $f(x)$ on the interval $[0, a]$ the inverse operation is

$$\int_0^a f(x)\, \mathrm{d}_q x = a(q^{-1/2} - q^{1/2}) \sum_{n=0}^{\infty} q^{(2n+1)/2} f(q^{(2n+1)/2} a) \tag{7}$$

and for the interval $[0, \infty)$

$$\int_0^\infty f(x)\, \mathrm{d}_q x = (q^{-1/2} - q^{1/2}) \sum_{n=-\infty}^{\infty} q^{(2n+1)/2} f(q^{(2n+1)/2}). \tag{8}$$

So

$$\frac{\mathrm{d}}{\mathrm{d}_q x}(ax^n) = a[n]x^{n-1} \tag{9}$$

$$\int ax^{n-1}\, \mathrm{d}_q x = \frac{1}{[n]} ax^n + \text{constant}. \tag{10}$$

Therefore, $(\mathrm{d}/\mathrm{d}_q x) e_q(ax) = a e_q(ax)$ and we have

$$\int e_q(ax)\, \mathrm{d}_q x = \frac{1}{a} e_q(ax) + \text{constant}. \tag{11}$$

From the definition of the q-derivative we derive the q-integration by parts formula:

$$\frac{\mathrm{d}}{\mathrm{d}_q x}(f(x)g(x)) = \frac{f(q^{-1/2}x)g(q^{-1/2}x) - f(q^{1/2}x)g(q^{1/2}x)}{q^{-1/2}x - q^{1/2}x}. \tag{12}$$

So

$$\frac{\mathrm{d}}{\mathrm{d}_q x}(f(x)g(x)) = \left(\frac{\mathrm{d}}{\mathrm{d}_q x} f(x)\right) g(q^{-1/2}x) + f(q^{1/2}x)\left(\frac{\mathrm{d}}{\mathrm{d}_q x} g(x)\right) \tag{13}$$

and

$$\int_0^a f(q^{1/2}x)\left(\frac{\mathrm{d}}{\mathrm{d}_q x} g(x)\right) \mathrm{d}_q x = f(x)g(x)\Big|_{x=0}^{x=a} - \int_0^a \left(\frac{\mathrm{d}}{\mathrm{d}_q x} f(x)\right) g(q^{-1/2}x)\, \mathrm{d}_q x. \tag{14}$$

Note that this result is not unique since we also have

$$\frac{\mathrm{d}}{\mathrm{d}_q x}(f(x)g(x)) = \left(\frac{\mathrm{d}}{\mathrm{d}_q x} g(x)\right) f(q^{-1/2}x) + g(q^{1/2}x)\left(\frac{\mathrm{d}}{\mathrm{d}_q x} f(x)\right). \tag{15}$$

Therefore

$$\int_0^a f(q^{-1/2}x)\left(\frac{\mathrm{d}}{\mathrm{d}_q x} g(x)\right) \mathrm{d}_q x = f(x)g(x)\Big|_{x=0}^{x=a} - \int_0^a \left(\frac{\mathrm{d}}{\mathrm{d}_q x} f(x)\right) g(q^{1/2}x)\, \mathrm{d}_q x \tag{16}$$

which is different from equation (14) above.

The resolution of unity for the q-analogue of the coherent states is based on a q-analogue of Euler's formula which we now derive.

We first define $-\zeta$ to be the largest zero of $e_q(x)$. Note that ζ is >0 and that as $q\to 1$, $e_q(x)\to\exp(x)$ and $-\zeta\to-\infty$. Whereas for $q\to 0$, $e_q(x)\to 1+x$ and $-\zeta\to -1$. We then redefine $e_q(x)$ to be

$$e_q(x)\equiv\sum_{n=0}^{\infty}\frac{x^n}{[n]!} \qquad \text{for } -\zeta<x \text{ and zero otherwise.} \tag{17}$$

In the same manner, we restrict the function $f(x)=x^n$ to be x^n for $-\zeta<x$ and zero otherwise. Then using q-integration by parts (14) we obtain

$$\int_0^{\zeta} e_q(-x)x^n\,\mathrm{d}_qx = q^{-n/2}[n]_q\int_0^{\zeta} e_q(-q^{-1/2}x)x^{n-1}\,\mathrm{d}_qx \tag{18}$$

and then

$$\begin{aligned}\int_0^{\zeta} e_q(-x)x^n\,\mathrm{d}_qx \\ &= (q^{-n/2}[n]_q q^0)(q^{-(n-1)/2}[n-1]_q q^{1/2})\ldots(q^{-1/2}[1]q^{(n-1)/2}) \\ &\quad\times\int_0^{\zeta} e_q(-q^{-n/2}x)\,\mathrm{d}_qx.\end{aligned} \tag{19}$$

Since

$$\int_0^{\zeta} e_q(-q^{-n/2}x)\,\mathrm{d}_qx = q^{n/2} \tag{20}$$

we see that all the qs cancel to leave

$$\int_0^{\zeta} e_q(-x)x^n\,\mathrm{d}_qx = [n][n-1][n-2]\ldots[1]=[n]!. \tag{21}$$

This is the q-analogue of Euler's formula for $\Gamma(x)$.

Note that starting with the left-hand side of (18) and using the other integration by parts formula (16), also yields (21).

The q-harmonic oscillator communication relations are

$$[a, a^\dagger] = aa^\dagger - q^{-1/2}a^\dagger a = q^{-N/2} \tag{22}$$

and

$$[N, a^\dagger] = a^\dagger \qquad [N, a] = -a.$$

Under the occupation number basis

$$a^\dagger|n\rangle = \sqrt{[n+1]}|n+1\rangle \tag{23}$$

$$a|n\rangle = \sqrt{[n]}|n-1\rangle \tag{24}$$

$$a|0\rangle = 0 \tag{25}$$

where $\langle m|n\rangle = \delta_{mn}$. The resolution of unity is written as

$$I = \sum_{n=0}^{\infty}|n\rangle\langle n|. \tag{26}$$

The q-coherent states are defined to be eigenstates of the operator a.

$$a|z\rangle_q = z|z\rangle_q. \tag{27}$$

For the normal coherent states, z is often a complex variable [27–29]. However, in general, z depends on other considerations, for example the dynamics of the physical system one is describing. For the resolution of unity for quantum groups it is natural to restrict $|z|$ such that $|z| \leqslant \sqrt{\zeta}$. From (27) we get

$$|z\rangle_q \equiv \langle 0|z\rangle_q \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}} |n\rangle. \tag{28}$$

Requiring $_q\langle z|z\rangle_q = 1$,

$$\langle 0|z\rangle_q = \exp(\mathrm{i}\phi) e_q(|z|^2)^{-1/2}. \tag{29}$$

Then choosing $\phi = 0$,

$$|z\rangle_q = N(z) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}} |n\rangle \tag{30}$$

where $N(z) = e_q(|z|^2)^{-1/2}$.

There exists a resolution of unity for the coherent states. The identity operator I can be written

$$I = \int |z\rangle_{qq}\langle z| \, \mathrm{d}\mu(z) \tag{31}$$

where

$$\mathrm{d}\mu(z) = \frac{1}{2\pi} e_q(|z|^2) e_q(-|z|^2) \, \mathrm{d}_q|z|^2 \, \mathrm{d}\theta. \tag{32}$$

Note that the integral over $\mathrm{d}\theta$ is a normal integration but the integration over $|z|^2$ is a q-integration. This result follows by

$$\int |z\rangle_{qq}\langle z| \, \mathrm{d}\mu(z)$$

$$= \frac{1}{2\pi} \int \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|z|^n |z^*|^m}{\sqrt{[n]!}\sqrt{[m]!}} e_q(-|z|^2) \, \mathrm{d}_q|z|^2 \times \int \exp(\mathrm{i}n\theta - \mathrm{i}m\theta) \, \mathrm{d}\theta |n\rangle\langle m| \tag{33}$$

$$= \frac{1}{2\pi} \int \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|z|^n |z^*|^m}{\sqrt{[n]!}\sqrt{[m]!}} e_q(-|z|^2) \, \mathrm{d}_q|z|^2 2\pi\delta_{mn} |n\rangle\langle m| \tag{34}$$

$$= \sum_{n=0}^{\infty} \frac{1}{[n]!} \int_0^{\zeta} x^n e_q(-x) \, \mathrm{d}_q x |n\rangle\langle n| \tag{35}$$

where $x = |z|^2$.

Then by the q-analogue of Euler's formula in the case of quantum groups

$$\int |z\rangle_{qq}\langle z| \, \mathrm{d}\mu(z) = \sum_{n=0}^{\infty} |n\rangle\langle n| = I. \tag{36}$$

So there exists a resolution of unity for the coherent states $|z\rangle_q$. The states with $|z|^2 = \zeta$ do not contribute.

For historical completeness, we note that using Jackson's definition of $[n]_J$ and one of his identities (2), Arik and Coon [24] obtained a similar relation fifteen years ago. However, the old [24] and new [12-14] q-analogue generalizations of the harmonic oscillator, the convergence properties of the $e_q(z)$, the q-integration identities, and the integration measures are non-trivially different.

For two arbitrary coherent states, $|\alpha\rangle_q$ and $|\beta\rangle_q$

$$\langle\alpha|\beta\rangle = N(\alpha)N(\beta)\sum_{m=0}^{\infty}\frac{(\alpha^*)^m}{\sqrt{[m]!}}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{[n]!}}\langle m|n\rangle \tag{37}$$

$$= N(\alpha)N(\beta)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha^*)^m}{\sqrt{[m]!}}\frac{\beta^n}{\sqrt{[n]!}}\delta_{mn} \tag{38}$$

$$= N(\alpha)N(\beta)\sum_{n=0}^{\infty}\frac{(\alpha^*\beta)^n}{[n]!}. \tag{39}$$

So

$$\langle\alpha|\beta\rangle = N(\alpha)N(\beta)e_q(\alpha^*\beta). \tag{40}$$

Since $|\alpha| \leqslant \sqrt{\zeta}$ and $|\beta| \leqslant \sqrt{\zeta}$, we have $|\alpha^*\beta| \leqslant \zeta$. In general $e_q(\alpha^*\beta) \neq 0$ and so arbitrary coherent states are not orthogonal.

As a result of the resolution of unity an arbitrary vector can be written

$$|\psi\rangle = \int |z\rangle_{qq}\langle z|\psi\rangle\,\mathrm{d}\mu(z). \tag{41}$$

Setting $|\psi\rangle = |\alpha\rangle_q$ an arbitrary coherent state, then

$$|\alpha\rangle_q = \int |z\rangle_{qq}\langle z|\alpha\rangle_q\,\mathrm{d}\mu(z) \tag{42}$$

so by the non-orthogonality of $|z\rangle_q$ and $|\alpha\rangle_q$, the q-analogue coherent states are linearly dependent. As a consequence, the q-analogue coherent states are not only complete but are actually overcomplete.

A more detailed treatment will be provided in a thesis of one of us [30].

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